Interior note

Transmitter:	For action:
A. LIEUTIER	

You will herewith find the document of analysis concerning the Package “Plate”, available in CAS.CADE and used in STRIM 6.2 (LONDON).

This package constitutes (with the Variational BATTEN, Minimal Curves Variation and Lissages) one of the first industrial applications of the variational or energy methods of generation of curves and surfaces.

The document refers to a “appendix 1” still non-existent which will indicate “bibliographical pointers” relating to certain allowed very technical results in the demonstration.
This appendix will be diffused soon (after passage to the library of the International Center of Mathematical Meetings).

The confidentiality relates to the chapter III which constitutes a description interns very detailed of Plate.
On the other hand, the chapters I, II and V could be used to carry out an external documentation (customer) Package.

A. LIEUTIER

I. Introduction

This document describes the algorithmic core “Plate”, so much from the point of view of the use (external interface, applications, limits and councils of use) that design (historical, mathematical model and description of the algorithm).
Extensions, generalization and another possibilities of algorithms in the same “vein” that Plate are simply evoked in this document. Their detailed study will be the another document object.

A. History and motivation

The exchange of topological models between applications which require different tolerances for coherence topology/geometry, poses well-known problems of the developers of interfaces of data exchange.
To cure this problem, it had been developed in STRIM an algorithm of “geometrical correction of the hulls”.
A key element of this algorithm consists in deforming a surface in order to make stick the image of a contour 2D in the parametric field, with a contour 3D given.
Traditional algebraic methods of filling (Hermitte, Coons.) apply when the constraints are fixed on isoparametric lines or “natural edges of squares”. The problem arising from the geometrical correction of the hulls does not satisfy this assumption.
The fact that there does not exist in general of polynomial or rational solution exact to the problem arising us led to be unaware of the algebraic representation of surfaces to pose the problem within a more general framework. The selected framework is that of the Partial derivative equations (EDP): the problem of the calculation of a “smooth” deformation enough satisfying constraints on a given parametric contour can be expressed as a EDP whose conditions at the borders are the imposed constraints.
The first idea consisted in using the EDP of the “thin sections” because it provides a rather pleasant physical analogy. The experiment, within the framework of the development of the geometrical correction of the hulls, then later, the theory, showed that, in general, this EDP led to solutions which can have specific C1 discontinuities and C2 linear which, in addition to the problems of surface quality are rather not very compatible with our tools for polynomial approximation.
One was thus brought to parameterize the order of the EDP, free to move away from the mechanical analogy. In addition the EDP is advantageously considered in its “weak” form, i.e. that the solution is expressed like a minimum under constraints.
The numerical method used can be regarded a method of the type “finite elements of border” or “integral method” or as a direct application of the theory of the variational splines.

In the context of the geometrical correction of the hulls, the imposed constraints were only G0 (the goal being to better connect the faces between them). The theory showed that the extension to constraints of a higher nature complicated the central algorithm little.
The needs for G2 connection gave the opportunity to us to realize, in technology CAS.CADE, a version of the algorithmic core able to manage G0 constraints, G1 and G2.
In addition, we carried out (cf document of Gerard Durand: filling with N dimensioned, GDD96347.DOC) a function of “filling to N dimensioned” in STRIM.
Lastly, 2 trainees of the ENSAM Lille, framed by Xavier Benvéniste, integrated Plate in Surfaces of Geom in order to “close” the fillets around the Vertexes.

A not yet planned but rather promising application would be an equivalent of the correction of the hulls in CAS.CADE. That poses other difficult problemsbut the Plate algorithm would allow a setting in G0 conformity, G1 or G2 of the geometry with orders of continuity desired along the edges.

B. the package Plate

Plate is a "package" with direction CAS.CADE, developed by the Scientific department and maintained by the team "AIX modeling".

Plate makes it possible to calculate a function defined on R2 values in R3.

This function is that which minimizes a quadratic "energy" by respecting linear constraints (in fact specified values).

Quadratic energy is:

$$E(\varphi) = \int_{R^2} (\nabla^m \varphi)^2 \qquad (1)$$

or the entirety m will be called "order" of the criterion thereafter . To avoid any ambiguity, one specifies the direction of the quantity under the sign of integration:

$$(\nabla^m \varphi)^2 = \sum_{i=0}^{m} \binom{i}{m} \cdot \left(\frac{\partial^m \varphi}{\partial^i u \partial^{m-i} v} \right)^2 \qquad (2)$$

In the expression above, one used the Anglo-Saxon notation for the coefficients of the binomial.

The square is a "scalar square", i.e. summons it squares of components X, y and Z. Thus*, E (ϑ)* can break up into three terms, for each component X, y and Z of φ.

If m=2, each one of these terms is the elastic energy of inflection linearized of a thin section whose Poisson's ratio is nulland whose field of normal displacement is the corresponding component of φ. From where the name, Plate, which means "English plate".

The constraints are specified values on the function and/or its derivative first and/or seconds. If the analogy of the plate is taken again, the constraints on the values correspond to conditions of support and those on the derivative first in conditions of embedding.

C. Strain of a surface and Gk stresses

In theory, it is possible to create a surface with Plate, since one has constraints on the position and the derivative in certain parameters (U, v). In practice, the simplest method to associate a parameter (U, v) a constraint is to project the constraint on an initial surface S. Ainsi, Plate will be generally used to calculate the deformation $\Delta\Sigma$ to be applied to a surface S so that S+DS satisfies a set of constraints given. In this context of use, Plate determines the Ck constraints automatically to apply to $\Delta\Sigma$ so that S+DS satisfies Gk constraints given.

D. Plate, esthetics and style

When initial surface S presents weak variations of curve and that its parameter setting is quite regular(it is the case for an initial surface plane, cylindrical, or quadric slightly curved.) and that the deformation $\Delta\Sigma$ is of low amplitude, the solution of the linearized EDP calculated by Plate is close to the solution to a EDP expressed starting from the purely geometrical characteristics of surface.

For example, when the order m is worth 3, the Plate solution is close to the surface which minimizes the quadratic average of the variation of curve.

Thus, always in this context of use, the Plate solution will lead to an overall harmonious evolution of the lines of light: the possible constraints G1 or G2 will not have a “local” effect as it is the case with certain algebraic methodsbut will be propagated “as well as possible” in order to minimize the total variation of curve (or variation of the variation for m=4.) on the whole of surface.
A Plate evolution able to minimize non-linear geometrical criteria, could preserve this property even when the assumptions on initial surface and the amplitude of the deformation are not checked any more.

II. External description of the Package Plate

A. *Ck constraints*

In this version one proposes only specific constraints (load with the user of the algorithm to sample the curves 2D on which he wants to impose constraints: to see the paragraph "applications").

A specific constraint is defined by:
- a point 2D (U, v) of gp_XY type,
- an order of derivation (iu, iv),
- a value of the type "gp_XYZ".

The order of derivation must check $0 \leq iu + iv \leq 2$.
The public interface of the specific "Forced" class is summarized in C++ with:

```
class Plate_PinpointConstraint
{
friend class Plate_Plate;
public:
Plate_PintpointConstraint (const gp_XY & UV, const gp_XYZ & Been worth,
const int iu = 0, const int iv = 0);
};
```

Note: in C++ really integrated in CAS.CADE, the int type and double in fact are declared respectively Standard_Integer and Standard_Real.

B. *Interface of the class Plate*

In the current version, one proposes the following services.
- The Plate object is initialized by its manufacturer.
- One can then add constraints to it.
- Constantly, one can ask to seek a solution.
- If the solution could be given one can then position on the solution and its derivative until order 2.

The public interface of the Plate class is summarized in C++ with:

```
class Plate_Plate
{
public:
Plate_Plate ();
~ Plate_Plate ();
void Load (const Plate_PintpointConstraint &);
void Load (const Plate_GtoCConstraint &);  //explained low
void SolveTI (const int order = 4, const double anisotropy = 1.);
Standard_Boolean IsDone () const;
gp_XYZ Evaluate (const gp_XY &) const;
gp_XYZ EvaluateDerivative (const gp_XY &, const int iu, const int iv) const;
};
```

1. Parameter “order” ofthe SolveTI method

It is at the time of the invocation of the **SolveTI** method that the **order** of the criterion (which is worth 4 per defect) is specified (it is thus possible, after “having charged” Plate by the invocation of the **Load** methods**,** to call upon several times the SolveTI method with different orders).

2. Parameter “anisotropy” ofthe SolveTI method

For questions of numerical conditioning and in order to make the result less depend on the parameterization of initial surface in the mode of use in deformation of surface, the energy given by (1) and (2) in fact is expressed compared to variables U and V, deduced respectively from U and v by a change of variable closely connected.
For a value 1 of the parameter “anisotropy” (default value), the change of variable is such as the smallest rectangle containing all them (U, V) is the square [0,1] X [0,1].

When surface presents an important lengthening in the direction of U or the v, one then advises with the user to indicate, by the means of the parameter “anisotropy”, the ratio dimension 3D in direction v (i.e length 3D of the Iso U) on dimension 3D in the direction U (i.e length 3D of the Iso v).
In this case, the change of variable imposes on the smallest rectangle containing all them (U, V), the same ratio “length” out of V over “length out of U”.
This ratio is called “**anisotropy**” because one can also interpret it like an anisotropy of “material constituting the plate” in the case m=2.

3. Methods IsDone, Evaluate and EvaluateDerivative

After the call to SolveTI, if the solution could not be found, for one of the reasons indicated to the following chapter, the IsDone method() turns over “False”. In the contrary case, this method turns over True and the methods Evaluate and EvaluateDerivative can be called upon.
This last method turns over the values of the derivative only until the total order (i.e nap of the orders of derivation out of U and v) equal to 2. Indeed the solution is not 3 times always everywhere derivable.

C. Expression of the constraints Gk (K =1,2)

The “pure” Plate algorithm accepts only linear constraints, in fact Ck constraints (the “PinPointConstraint” described above).
However, a particularly important case of use of the algorithm is the calculation of the deformation of a surface under pressures Gk (K =1 or 2).
It is supposed that one has a function S of two variables to values in R3. One wants to use the Plate algorithm to calculate the deformation $\Delta\Sigma$ to be added to S so that S+DS checks constraints G1 or G2.
The constraints G1 (or G2) in a given parameter are expressed by the derivative until order 1 (or 2) of a function T with which one wishes to have a contact of order 1 (or 2).
Let us call DS and D2S the derivative first and second of S, surface to be deformed, and DT and D2T those of the function T with which one wants to ensure a contact of order 1 or 2.
In order to express a G1 constraint, we should DS and DT, and in order to express a constraint G2, it is necessary DS, D2S, DT and D2T for us.

One calls D1 (and D2 respectively) types of structures containing of the values of derived first (and seconds) of a function of two variables to values of the gp_XYZ type (D1 contains 2 “gp_XYZ” and D2 contains 3 of them).

One then proposes the utilities according to:

```
class Plate_D1
{
	friend class Plate_GtoCConstraint;
public:
	Plate_D1 (const gp_XYZ & of, const gp_XYZ & FD);
};

class Plate_D2
{
	friend class Plate_GtoCConstraint;
public:
	Plate_D2 (const gp_XYZ & duu, const gp_XYZ & duv, const gp_XYZ & dvv);
};

class Plate_GtoCConstraint
{
	friend class Plate_Plate;
public:
	Plate_GtoCConstraint (D1 &DS, D1 &DT); //G1 constraint
	Plate_GtoCConstraint (D1 &DS, D2 &D2S, D1 &DT, D2 &D2T); //G1 + G2 constraint
};
```

According to whether one expresses a constraint of order 1 or 2 this type of constraint corresponds to 2 or 5 “PinPointConstraint”.
Notice that to in no case the G0 constraintis not imposed. One will have generally to thus add to this end a “PinpointConstraint” of order 0 to the same parameter.

There is a priori an infinity of Ck constraints which ensure Gk compatibility. One chooses that which impacts less the parameterization of initial surface. That is explained in detail in the following chapter (III).

Note:

- It is imperative that DS and DT are regular (i.e generates a vector space of dimension 2 or, in other terms, that the vector product of the derivative partial is nonnull).
- It is also essential that the normals with DS and DT (i.e caused vector products) are not orthogonal and the result will have little chance to be "aesthetically" acceptable if the angle between these normals is important (by "important" one understands higher than 60 degrees for example).
- There are on the other hand no limitation on the values of the derived seconds, D2S and D2T.

One gives in the chapter "limitations and council of uses" some methods to circumvent these limitations.

D. Possible extensions of the interface

One can imagine the following extensions to the Plate interface.

- Possibility of adding specific "loadings". The term of "loading" means in the context of the analogy "thin section": to apply a "specific force", is equivalent to impose a déplaceme, NT at the same point but, instead of controlling the value of displacement, one controls the value of the force of reaction.

- Possibility of removing constraints and loadings by handling them by their handle or their name.

- Possibility of varying the vectorial intensities of the constraints or the loadings without modifying the site of it, which can make it possible to calculate a solution quickly (only the second member of the linear system changes), this for a possible use in "dynamic action".

- Possibility of having various methods of resolution (e.g. finite elements). The interest of a Plate version based on the finite element method is to be well conditioned numerically even for "difficult" cases and to give directly in result a surface spline (what avoids having to operate a polynomial approximation).

- Possibility of adding linear constraints (G0, G1 or G2) checked to a given precision. In this case, the algorithm would sample the lines to create specific constraints. Sampling "would be refined" until the error is checked. In the current version, this treatment must be carried out by the user of the algorithm (see Applications paragraph).
- Possibility of applying constant or variable pressures. A pressure is a loading which specific but is not distributed on a zone. That can make it possible to deform surface interactivement. The localization or the "form" of the deformation is then a function of the distribution of the selected pressure.

The interface proposed is compatible with all these extensions (in the direction where the addition as of the these extensions would not impact the source code using the first version).

III. Description interns of Plate

In the continuation, we give a detailed explanation of the algorithm, by admitting certain theoretical results. Chapter III, A remains however very technical and it is possible to pass directly following the document by admitting the results simply concerning:

- the condition of unicity,
- the expression of the solution,
- the equilibrium condition of the generalized moments.

A. Mathematical model of the principal algorithm

A completely rigorous talk of the validity of the algorithm is difficult, request to place itself in the context of the functional analysis (construction of specific spaces of Hilbert, distributions.) and of the theory of Splines variational (cf for example the book of P.J. Laurent at Hermann, collection “teaching of sciences” Approximation and optimization).
In paper ISATA 95, G.Durand, A. Lieutier, A. Massabo, “Incompatible Accuracy Modelers: Did Are Complex Date Exchanges Possibles?”…, one gives the broad outline of a proof of convergence of the solution “discretized” towards the solution corresponding to linear constraints, while referring to work of V.Vassilenko.
Chapter III of the book of Marc Atteia, “Hilbertian Kernels and Spline Functions” (Editions North Holland 1992) presents the functions of “Shoenberg” whose Plate is an example. The last example taken by Marc Atteia (page 174) is related exactly to Plate, if one imposes only C0 constraints. The reader who wishes to know in detail the theoretical framework necessary to show all the results will be able to refer to it.

1. Independence of the three components

The problem defined by the Ck constraints can break up into three independent problems on components X, y and Z of φ. Indeed:

- energy can be expressed as the sum of three terms, each term relating to only one co-ordinate, X, y or Z, of φ, the sought function,
- each Ck constraint is equivalent to three constraints, carrying each one on only one of the components of φ.

In other words, neither the expression of energy, nor that of the Ck constraints create coupling between the components of φ.

Thus, φ is solution of the problem if and only if each one of its components minimizes the corresponding component of energy by respecting the corresponding component of each constraint.

The constraints Gk, they, create a coupling between the components and are in general not linear. It is one of the reasons for which one “transforms them” into Ck constraints by the method described into III, C.

In the continuation, one concentrates on the problem concerning only one component of φ, called F.

2. Notations and reformulation of the problem

For two functions F and G of L2 (R2), one notes <f, g> their product scalar in L2 (R2), i.e.:

$$\langle f, g\rangle = \int_{R^2} f \cdot g$$

That is to say m a positive entirety, equal to or higher than 2.
One notes the ***Xm*** space of the functions whose derivative mième is in L2 (R2).
It will be admitted that:

$$Xm \subset cm2\ (R2)$$

One can notice that each component of φ is in Xm if and only if the energy of φ, defined by (1) and (2) is finished.
Xm is thus the space in which one seeks the solution of the problem (for only one component of φ).

For two functions *F* and *G* of ***Xm***, one notes *<f, g>m the* scalar product of their derivative mièmes, i.e.:

$$\langle f, g\rangle_m = \langle \nabla^m f, \nabla^m g\rangle = \int_{R^2} \sum_{i=0}^{m} \binom{i}{m} \frac{\partial^m f}{\partial^i u \partial^{m-i} v} \frac{\partial^m g}{\partial^i u \partial^{m-i} v}$$

A constraint on a function F in a parameter (U, v) is expressed by:

$$\frac{\partial^{i+j} f}{\partial^i u \partial^j v}(u, v) = C$$

and is thus defined by the data of a couple of realities (U, v), of a couple of entireties (I, J) defining the order of derivation and of real C defining the specified value in (U, v).
One supposes given N such constraints.

One can now reformulate the problem (for a component).
Thereafter one will call "problem" **P**, the following problem:

$$\begin{array}{l} f \in X_m \\ \quad f \ \text{ minimise } \langle f, f\rangle_m \\ \text{avec la contrainte :} \\ \quad \forall k = 1, n \quad \dfrac{\partial^{i_k + j_k} f}{\partial^{i_k} u \partial^{j_k} v}(u_k, v_k) = C_k \\ \text{où} \\ \quad \forall k = 1, n \quad 0 \le i_k + j_k \le m - 2 \end{array}$$

It is supposed, in the expression of P, that for two entireties K and k' different, one forever (the U.K., vk, ik, jk) = (uk', vk', ik', jk') in order to avoid having two equivalent or contradictory constraints (according to whether the values of corresponding Ck and Ck' are equal or different).

One notes ***Pm-1the*** whole of the polynomials of two variables of degree (with more) M-1.
It is known that ***Pm-1*** is dimension m (m+1)/2 and that:

$$Pm\text{-}1 \subset Xm$$

and one checks without difficulty that:

$$<f, f>m = 0 \Leftrightarrow F \in Pm\text{-}1$$

In the case m=2, corresponding to the linearized thin sections, this result is in conformity with the intuition: only the plans, (of equation Z = A.U + b.v + C, i.e polynomial of total degree 1), induce an elastic energy of null inflection.
In the following chapter, we go to express the solution of the problem P in the form of the sum of a particular solution and a polynomial of ***Pm-1***.

3. Condition of unicity of the solution

One will admit without demonstration that the problem P admits at least a solution in Xm. The demonstration is based for example on the theorem of Lax-Milgram (cf H. BREZIS, Analyze functional, Théorie and applications, 1983 at MASSON, page 84) after having built "the good" space of Hilbert. One will be able to also consult Atteia 92 (Hilbertian Kernels and Spline Functions, Mr. ATTEIA, NORTH-HOLLAND).

On the other hand one proposes to show that the following condition is a condition necessary and sufficient for the unicity of the solution.

There is not polynomial not no one of degree M-1 which cancels N forced.

Condition U (second expression):

$$\left\{ p \in P_{m-1} \; ; \forall k = 1, n \quad \frac{\partial^{i_k + j_k} p}{\partial^{i_k} u \partial^{j_k} v}(u_k, v_k) = 0 \right\} = \{0\}$$

One can give an intuitive interpretation of this condition in the case *m=2* and *ik=jk=0*, i.e. that one considers only C0 constraints. In this case the condition is expressed by: any polynomial of degree 1 which is cancelled in each *(the U.K., vk)* is null. What comes down to saying that there are at least 3 *(the U.K., vk)* not aligned. That still corresponds to the intuition, one needs at least 3 feet not aligned to support a table.

Demonstration:

The condition is necessary. Indeed, if it is not checked, that means that there is a polynomial not no ***p*** of degree *M-1* such as:

$$\forall k = 1, n \quad \frac{\partial^{i_k + j_k} p}{\partial^{i_k} u \partial^{j_k} v}(u_k, v_k) = 0$$

Either F a solution of the P. problem One then checks simply that:

- ***<f+p, f+p>m = <f, f>m***
- ***f+p*** satisfies the constraints

f+p is thus still solution of **P** and the minimum is not single.

Reciprocally, if there are two solutions ***F*** and ***G*** of **P**, with ***F*** different of ***G***, then, for all real T, ***f+t (g-f)*** checks the constraints (bus ***F*** checks the contaraintes and ***(f-g)*** "cancels them") and one a:

$$<f+t\,(g-f), f+t\,(g-f)>m = <f, f>m + 2t <f, g-f>m + t2 <g-f, g-f>m$$

this is a polynomial of degree 2 in T which, according to the assumption on F and G reaches its minimum in T =0 and T =1.
Therefore, this polynomial is necessarily degree 0.
Thus, one a:

$$<g-f, g-f>m = 0$$

what, as one saw with III, A, 2, involves:

$$g-f \in Pm-1$$

The fact that ***G*** and ***F*** satisfy the constraints shows that ***g-f*** is the polynomial not no one which invalidates the condition U.
End of the demonstration.

4. Expression of the solution and "moments generalized"

One will apply the principle of the "multipliers of Lagrange" to the P. problem Although it is not the traditional method to arrive at the result, it will be more intuitive for the readers who have experience of optimization under constraints.
However, the reader is supposed to have some concepts on the distributions, how they are derived, and basic properties on the product of convolution between distributions. For a talk of these concepts the reader will be able to consult for example "J.Bass, course of mathematics", Tome III, at Masson.

The principle of Lagrange applied to the problem P gives a linear relation between the derivative of energy and the derivative of the constraints. This equation between "linear forms on ***Xm***" is then applied to **Pm-1**, which gives the equilibrium equation of the generalized moments, and to the space of the functions tests of the distributions, which gives an expression of the solution.

The derivative out of F of an application of Xm in R, when it exists, is a continuous linear form on Xm.

The derivative of *<f, f>m*, out of F, is the linear form which with any *G* of *Xm* fact of corresponding *2<f, g>m.*
This derivative will be noted:

$$2 <f, \bullet >m.$$

We now will express the derivative of the constraints using distributions.
*One will admitwhom the assumption i+j $\leq$ m2 involves that the application, definite on **Xm**, which with **F** makes correspond:*

$$\frac{\partial^{i+j} f}{\partial^i u \partial^j v}(u,v)$$

is continuous.
To reduce the notation, this last expression will be noted:

$$F^{(I,\ J)}\ (U,\ v)$$

Definition of the distribution of Dirac, one can write this expression:

$$<d_{(U,\ v)},\ F^{(I,\ J)}>$$

where $\delta_{(U,\ v)}$ is the distribution of Dirac "centered" in (U, v).
What is still written, according to the definition of derived from a distribution:

$$<(-1)^{i+j}\ \delta_{(U,\ v)}{}^{(I,\ J)},\ f>$$

(The reader which is not familiar with the distributions will be able to be convinced some after i+j integrations by parts).
The constraints being linear, their derivative are constant and equal to the linear forms defining the constraints themselves.
Thus, the derivative of the forced kème is written:

$$\left\langle (-1)^{i_k + j_k}\, \delta_{(u_k, v_k)}{}^{(i_k, j_k)}, \bullet \right\rangle$$

One is able now to write the condition of Lagrange for a minimum under constraints. Let us recall that this principle stated as follows:

If E and Ci$_{,\ i=1,\ N}$, are derivable functions and if X is a local minimum of E under constraints Ci (X) = Bi, then, there are $\Lambda\iota$ coefficients such as:

$$\frac{\partial E}{\partial X} = \sum_i \Lambda_i \frac{\partial C_i}{\partial X}$$

Let us express this principle in our case:

$$\exists \lambda_{1,}, \ldots, \lambda_n \in R$$

$$\langle f, \bullet \rangle_m = \sum_{k=1}^{n} \lambda_k \left\langle (-1)^{i_k + j_k}\, \delta_{(u_k, v_k)}{}^{(i_k, j_k)}, \bullet \right\rangle$$

The symbol $\bullet$ mean that the expression is true if one replaces it by any function of Xm.
The report/ratio 2 which appears in the expression of derived from <f, f>m was omitted because it can "be absorbed" by the coefficients $\lambda\kappa$.
One can also interpret physically the multipliers of Lagrange $\lambda\kappa$ like the specific forces and specific moments of reaction (generalized moments if i+j>1) corresponding to the constraints.

We consider the consequences of this expression if • respectively traverses the two subspaces following of Xm:

- *Pm-1*
- *𝒟, the space of the functions "test" of the distributions, i.e. the whole of the C¥ functions to compact support.*

Application to Pm-1:

In the first case, the member of left of the equality is null and the condition of Lagrange provides us the condition "of BALANCE of the GENERALIZED MOMENTS" (EMG) which one will express in two equivalent forms, EMG1 and EMG2.

Form EMG1is:

$$\forall p \in P_{m-1}$$
$$\left\langle \sum_{k=1}^{n} \lambda_k (-1)^{i_k + j_k} \delta_{(u_k, v_k)}{}^{(i_k, j_k)}, p \right\rangle = 0$$

What one can also write:

$$\forall p \in P_{m-1}$$
$$\sum_{k=1}^{n} \lambda_k p^{(i_k, j_k)}(u_k, v_k) = 0$$

There is equivalence between the nullity of a linear form on a vector space and the fact that this linear form is null for each vector of a base of the vector space. The base of the uivj is chosen, where i+j lies between 0 and M-1.

One notes pi, J, i', l it derived from order (i', l) of uivj.

One a:

$$p_{i,j,i',j'}(u,v) = \begin{cases} 0 \text{ si } i' > i \text{ ou } j' > j \\ \dfrac{i!\,j!}{(i-i')!(j-j')!} u^{i-i'} v^{j-j'} \text{ sinon} \end{cases}$$

One can then write EMG2:

$$\boxed{\begin{array}{l} \forall i, j; \quad 0 \le i + j \le m-1 \\ \displaystyle\sum_{k=1}^{n} \lambda_k p_{i,j,i_k,j_k}(u_k, v_k) = 0 \end{array}}$$

Application to 𝒟:

If ψ is related to 𝒟, then:

$$\langle f, \psi \rangle_m = \langle \nabla^m f, \nabla^m \psi \rangle = (-1)^m \langle \Delta^m f, \psi \rangle$$

If F were 2m time derivable, this equality would rise from m integrations by parts. In fact this relation is always true but the Laplacian with the power m of F must be included/understood "within the meaning of the distributions".
Thus, the condition of Lagrange makes it possible to write the following equality, which it is necessary to include/understand like an equation between distributions, that one will call ***ED****:*

$$(-1)^m \Delta^m f = \sum_{k=1}^{n} \lambda_k (-1)^{i_k + j_k} \delta_{(u_k, v_k)}^{(i_k, j_k)}$$

That is to say EM the function defined on R2 by:

$$E_m(u,v) = \frac{1}{4\pi}(2^{m-1}(m-1)!)^{-2}(u^2+v^2)^{m-1} Log(u^2+v^2)$$

It will be admittedthat EM is solution elementary of the $\Delta\mu$ *operator, i.e. that:*

$$\Delta^m E_m = \delta$$

Thus, for any distribution with compact support D*, one can write, by noting* $*$ *produces it convolution (between distributions):*

$$\Delta^m (E_m * d) = \Delta^m E_m * d = \delta * d = d$$

In particular, let us consider ***F*** *defined by:*

$$F = E_m * (-1)^m \sum_{k=1}^{n} \lambda_k (-1)^{i_k + j_k} \delta_{(u_k, v_k)}^{(i_k, j_k)}$$

F is well solution of the equation between distributions above (ED) whose unknown factor is F. It is to some extent about a "particular solution".

Although EM is not in Xm, it will be admitted that the fact that the $\lambda\kappa$ *check the balance of the generalized moments involves that the distribution F above is in Xm. We do not know a simple demonstration of this proposal.*

The function F sought and F are solutions of ED and belong to Xm, one can thus write:

$$(f - F) \in X_m \text{ et } \Delta^{\mathrm{m}}(f - F) = 0$$

One can then apply the first part of proposal 4.3 of Atteia92 and one a:

$$(f - F) \in P_{m-1}$$

Thus, F is the sum of F and a polynomial of degree M-1.
One can write the form of the solution F of the problem P:

$$f = (-1)^m \sum_{k=1}^{n} \lambda_k (-1)^{i_k + j_k} E_m * \delta_{(u_k, v_k)}^{(i_k, j_k)} + p_{m-1}$$

where pm-1 is a polynomial of degree M-1.
It is possible to clarify the product of convolution:

$$E_m * \delta_{(u_k, v_k)}^{(i_k, j_k)}(u,v) = E_m^{(i_k, j_k)}(u - u_k, v - v_k)$$

As follows:
there is N real $\lambda\kappa$ *and m (m+1)/2 realities cij such as the solution F of P is form:*

$$f(u,v) = (-1)^m \sum_{k=1}^{n} \lambda_k (-1)^{i_k+j_k} E_m^{(i_k,j_k)}(u-u_k, v-v_k) + \sum_{0\le i+j\le m-1} c_{ij} u^i v^j$$

Τηε λκ check the equilibrium condition of generalized moments EMG2 above.
The theorem of existence of the solution of P, the condition U of unicity, the equilibrium condition of generalized moments EMG2 and the form of the solution F given above will enable us to build the numerical method.

B. Construction of the linear system

We will rewrite the results of the preceding chapter in a form adapted better to the numerical resolution.
All the constant coefficients which are in factor of the λκ can be omitted (they can be absorbed by the λκ). Thus one will omit the term (- 1) ^{m} which appears in the expression of the solution and one removes the constant coefficient which appears in the expression of EM.
One considers the function ℰℳ , "elementary solution with a multiplicative factor near":

$$\mathcal{E}_m(u,v) = (u^2+v^2)^{m-1} \cdot Log(u^2+v^2)$$

The solution F is written then:

$$f(u,v) = \sum_{k=1}^{n} \lambda_k (-1)^{i_k+j_k} \mathcal{E}_m^{(i_k,j_k)}(u-u_k, v-v_k) + \sum_{0\le i+j\le m-1} c_{ij} u^i v^j$$

It would be possible to remove the term $(-1)^{i_k+j_k}$ (it could also be absorbed by the λκ), but that would make lose the symmetry of the matrix of the linear system.

We have N unknown factors λκ and m (m+1)/2 unknown factors cij, are n+ m (m+1)/2 unknown factors.
We obtain N equations linear while expressing that F satisfies the linear constraints and m (m+1)/2 equations by expressing the balance of the moments generalized in form EMG2.

N forced:

$$\forall k', 1 \le k' \le n, \quad f^{(i_{k'},j_{k'})}(u_{k'}, v_{k'}) = C_{k'}$$

express themselves:

$$\forall k', 1 \le k' \le n,$$

$$\sum_{k=1}^{n} \lambda_k (-1)^{i_k+j_k} \mathcal{E}_m^{(i_k+i_{k'}, j_k+j_{k'})}(u_{k'}-u_k, v_{k'}-v_k) + \sum_{0\le i+j\le m-1} c_{ij} p_{i,j,i_{k'},j_{k'}}(u_{k'}, v_{k'})$$

where pi, J, i', I (U, v) is the students'rag procession defined in the preceding chapter.
Thus, if one poses:

$$
\boxed{\begin{array}{l} a_{kk'} = (-1)^{i_k + j_k} \mathcal{E}_m^{(i_k + i_{k'}, j_k + j_{k'})}(u_{k'} - u_k, v_{k'} - v_k) \\ b_{ijk'} = p_{i,j,i_{k'},j_{k'}}(u_{k'}, v_{k'}) \end{array}}
$$

EMG2 is expressed:

$$
\forall i', j', 0 \le i' + j' \le m - 1,
$$

$$
\sum_{k=1}^{n} \lambda_k b_{i'j'k} = 0
$$

One can then write the linear system in matric form:

$$
\boxed{\begin{pmatrix} a_{kk'} & b_{ijk'} \\ b_{i'j'k} & 0 \end{pmatrix} \begin{pmatrix} \lambda_k \\ c_{ij} \end{pmatrix} = \begin{pmatrix} C_{k'} \\ 0 \end{pmatrix}}
$$

It rises immediately from the shape of $\mathcal{EM}$ that:

$$
\mathcal{EM}(-U, -v) = \mathcal{EM}(U, v)
$$

One deduces some:

$$
\mathcal{EM}^{(I, J)}(-U, -v) = (-1)^{i+j} \mathcal{EM}^{(I, J)}(U, v)
$$

As follows:

$$
\begin{aligned}
a_{kk'} &= (-1)^{i_k + j_k} \mathcal{E}_m^{(i_k + i_{k'}, j_k + j_{k'})}(u_{k'} - u_k, v_{k'} - v_k) \\
&= (-1)^{i_k + j_k} (-1)^{i_k + i_{k'} + j_k + j_{k'}} \mathcal{E}_m^{(i_k + i_{k'}, j_k + j_{k'})}(u_k - u_{k'}, v_k - v_{k'}) \\
&= (-1)^{i_{k'} + j_{k'}} \mathcal{E}_m^{(i_k + i_{k'}, j_k + j_{k'})}(u_k - u_{k'}, v_k - v_{k'}) = a_{k'k}
\end{aligned}
$$

One deduces from it that the matrix of the linear system is symmetrical.

In addition, if the condition of unicity U, given to the preceding chapter is checked the existence and the unicity of a solution which checks the linear system involves that this linear system is theoretically not singular (what does not guarantee that it cannot be badly conditioned in certain cases). Possible not a respect of the condition U is detected by the method of resolution of the linear system which diagnoses a singular system.

If one calls Oc the total order of derivation (i.e nap of the orders out of U and v) on the imposed constraints and Off the order of the derivative which one wishes to calculate on F, the fact that function $\mathcal{EM}$ is C2m-3 involves the following inequalities, which rise respectively from the form of akk' and that of F (U, v):

$$
2\,Oc \le 2m-3
$$

$$
Oc + Of \le 2m-3
$$

The first inequality is equivalent to the condition Oc$\le$ m2, expressed in the form ik+jk $\le$ m2 used in the ideal model.

From a practical point of view, one chose to limit oneself to:

$$
Oc \le 2
$$

$$
Off \le 2
$$

Thus, since one has m ≥ 2, τηε condition:

$$Oc \leq m2$$

is sufficient.

This practical limitation could be raised but the experiment shows that, when m is larger than 6, the system is generally badly conditioned (of course that is progressive, conditioning is sometimes already bad for m=5.) and the fact of imposing constraints of a nature higher than 3 should also harm conditioning. If one really wishes to impose constraints of a higher nature, perhaps it will be necessary to await the writing of a "Plate" using an algorithm of the type "finite element", only means, to our knowledge, to push back very far the problems of numerical conditioning.

In term of programming, the constitution of the linear system as well as the calculation of F (U, v) and its derivative once the coefficients known λκ and cij does not pose any problem provided that one has the functions:

- $pi_{, J, i', I}$ *(U, v),* with: $0 \leq \iota+\varphi \leq \mu-1$ and 0£ $\iota\ni+\varphi\ni\leq 2$
- $\mathcal{EM}^{(I, J)}$ *(U, v),* with: $0 \leq \iota+\varphi \leq 4$

These functions are private methods of Plate called respectively Polm and SolEm. Polm does not pose a problem.

On the other hand, the formal calculation and the coding of derived until order 4 from $\mathcal{EM}$*, which constitute the body of SolEm would have constituted a nightmare if one had not been able to generate the C++ code with Mathématica.*

C. Expression of the Ck constraints according to the Gk constraints

In the mode of use in "deformation" (only used today), a "initial surface" S is given and the function calculated by Plate is a function ΔΣ (here Δ does not mean any more Laplacien but "deformation") such as S+DS satisfies constraints G1 or G2.

The GtoCConstraint class has the aim of calculating the Ck constraints to be forced on ΔΣ to aprtir Gk constraints on S+DS.

The data, for a parameter (U, v) given are the derivative first (and seconds for G2) of S and T, a surface with which one want to impose the G1 contact (and possibly G2).

One takes again notation DS, D2S, DT and D2T for the derivative first and seconds of S and T.

The G1 constraints could have been defined by a normal vector. In fact, if one imposes a constraint unconstrained G1 G2, the couple of vectors DT intervenes only through vector product DT1 X DT2. One could besides without no difficulty of adding a manufacturer of GtoCConstraint starting from DS and a normal vector in the case of forced G1.

However, for the G2 constraints, it is unnecessarily complicated, as well for the user of the algorithm as for GtoCConstraint to use a canonical representation of the curve of the type "principal direction + principal curves".

The method used extracts "naturally" geometrical information from the parametric data.

It is supposed that S and T are regular and, more precisely than the projection of DS on the level generated by DT is of row 2, which is expressed:

"(DS1 xDS2) •(DT1XDT2) is nonnull"

The “singular” cases where it is not checked are not treated. In the current Plate version, the constraint, in this case, is simply ignored. If that is necessary, one will be able to add a method “IsDone” to the GtoCConstraint class to indicate to appealing that the constraint is invalid.

It is wished that S+DS, in a given parameter, have a contact of order G1 or G2 with T in a given parameter. There is an infinity of corrections DDS (and D2DS) to bring to DS (and D2S) which satisfy this condition.
On the other hand there is only one choice of DDS (and D2DS) which in addition to satisfying the contact of order 1 (or 2), makes DDS (and D2DS) orthogonal with DS.
It is this correction which one chooses because it has the advantage of not disturbing S in the tangential direction.
For example, in the case enough current where S is a plan, that guarantees that S+ $\Delta\Sigma$ does not present any car-intersection. Indeed, the Plate solution depends linearly on the constraints C0, C1 and C2 imposed. Thus, if all these constraints are colinéaires with normal with plan (it is also the case of the C0 constraints if the parameters (U, v) of each points are obtained by orthogonal projection), the solution $\Delta\Sigma$ is still orthogonal in the plan (each component X, y, and Z of the solution are in the same report/ratio as components X, y and Z of the constraints). Thus, if there is S (U, v) + $\Delta\Sigma$ (U, v) = S (u', v') + $\Delta\Sigma$ (u', v'), one can break up the equality according to his component in the plan and his orthogonal component: unless the plan car-is not intersected final surface cannot car-be intersected.

1. Calculation of the C1 constraints on $\Delta\Sigma$ induced by the G1 constraints on S+DS

One notes $S_{,I}$ for i=1,2 (resp. $T_{,I}$), the derivative first of S (resp. T). This notation differs from that of the preceding chapter but is adapted better to calculation indiciel.
One notes $\Delta\Sigma_{,I}$ the corrections to be brought to $S_{,I}$, which will constitute the C1 constraints for Plate.
One considers the normalized normals with S and T, noted NS and NT:

$$n_S = \frac{S_{,1} \times S_{,2}}{\left\| S_{,1} \times S_{,2} \right\|} \text{ et } n_T = \frac{T_{,1} \times T_{,2}}{\left\| T_{,1} \times T_{,2} \right\|}$$

It is wished that:

$$\exists \alpha_i \in \mathbf{R},\ \Delta S_{,i} = \alpha_i n_S \text{ et } (S_{,i} + \Delta S_{,i}) \cdot n_T = 0$$

What gives immediately:

$$\boxed{\alpha_i = \frac{-S_{,i} \cdot n_T}{n_S \cdot n_T}}$$

And the C1 constraints deduced on $\Delta\Sigma$ are:

$$\boxed{\Delta S_{,i} = \alpha_i n_S}$$

2. Calculation of the C2 constraints on $\Delta\Sigma$ induced by the G2 constraints on S+DS

To reduce the writing, one invites, $_{I}$ the derivative first of S corrected (by the corrections calculated above), i.e.:

$$S'_{,i} = S_{,i} + \Delta S_{,i}$$

According to the initial assumption, $T_{,i}$ and$_{,i}$ constitute all two a base of the imposed tangent plan, i.e. the orthogonal plan with NT.

There is thus a matrix 2x2, which one will note Aik, such as:

$$S'_{,i} = \sum_k A_i^k T_{,k}$$

Aik is calculated simply according to the relation:

$$\boxed{S'_{,j} \cdot S'_{,i} = \sum_k A_i^k S'_{,j} \cdot T_{,k}}$$

Indeed, this relation gives the coefficients of Aik as solution of a linear system of order 2.
The Aik matrix can be interpreted as a change of variable in the parameterization of T in the vicinity of the point which interests us such as, with this new parameterization, the derivative first of T and of S+DS, at the point considered, coincide.
That can be written at the point concerned:

$$\frac{\partial T}{\partial u'_i} = \sum_k \frac{\partial u_k}{\partial u'_i} \frac{\partial T}{\partial u_k} = \sum_k A_i^k \frac{\partial T}{\partial u_k} = S'_i$$

Thus, if surface T is reparamétrée by the u'_i, its derivative first coincide with those of S+ $\Delta\Sigma$. One can suppose (to simplify calculations but that the result would not change) that, at the point considered, the derivative second of u' compared to U are null or although the change of variable is linear.

One will admit the following proposal:

Are S1 and S2 two parameterized surfaces which coincide in a given point, presumedly regular and twice derivable in this point. One supposes moreover than their derivative first coincide in this point. then:
S1 and S2 have a contact of order 2 in this point if and only if the normal component of their derivative second are equal.

One can apply this property to the couple of surface S+ $\Delta\Sigma$ and T reparamétré by u'_i.
One thus uses the following property to characterize a contact of order 2:

$$n_T \cdot \left(S_{,ij} + \Delta S_{,ij} - \sum_{k,l} A_i^k A_j^l T_{,kl} \right) = 0$$

The condition of colinearity of $\Delta\Sigma_{,ij}$ with NS gives us:

$$\exists \beta_{ij} \in \mathbf{R}, \quad \Delta S_{,ij} = \beta_{ij} n_S$$

After substitution under the preceding condition, that gives:

$$\boxed{\beta_{ij} = \frac{n_T \cdot \left(\sum_{k,l} A_i^k A_j^l T_{,kl} - S_{,ij} \right)}{n_T \cdot n_S}}$$

And the C2 constraints deduced on $\Delta\Sigma$ are:

$$\boxed{\Delta S_{,ij} = \beta_{ij} n_S}$$

IV. Applications

A. Applications already carried out

1. Algorithm of filling to N with dimensions

This application is described in detail in GDD96 347.DOC.
A result of a good esthetic quality is obtained when there are no problems of incompatibility between the constraints and that there is an initial surface simple (plan or quadric) adapted well. Adapted well means that the required deformation modifies little the parametric distribution (or the metric one) of surface.

The response times and the number of generated squares, rather important, are not a handicap in a context of use in "surface complex" for which alternative, manual or non-existent methods are in any event not competitive.

2. Setting in G2 continuity of H-PRISM.

For this problem, the solutions of the type "zone of transition" or "filling edge + interior" do not seem satisfactory, in particular on an esthetic level.
One can use the Plate algorithm to calculate a correction DS of H-PRISM S by imposing C0 constraints and constraints of the type "GtoCConstraint".
This method is likely good aesthetically to give results (HighLight) acceptable in the "nice" cases (i.e tended well and regular) which are generally those of the designers.
On the other hand:

- *this method is inoperative on difficult cases (strong incompatibility of constraints to the corners, "tortured" constraints, i.e high frequencies),*
- *if the defect of G2 continuity to be corrected is important, the result can move away from initial H-PRISM of an unforeseeable distance intuitively by the user.*

However, on the realistic cases, the correction seems to give satisfaction to the customer.

B. some future applications

1. Leave on adjacent top with N edges

This problem can be brought back to a filling to N (or more) with dimensions (cf the letter of the DS number 1). The method consisting in operating a polynomial approximation of the Plate solution with the current algorithm is available short-term but is not very economic in computing times and will lead to failures in the twisted cases (impossibility of obtaining the precision with a reasonable number of squares).

It will be necessary to await the availability of a method of the type "Ritz" (cf below) to cure these foreseeable problems.

2. Gn correction of the hulls

Plate should make it possible to develop in CAS.CADE the equivalent of the geometrical correction of the hulls of STRIM with conservation or improvement of continuities G1 and G2. Of course this development is rather important because it would be necessary to take into account two improvements compared to the operator available in STRIM:

- *correction/relieving of the curves 2D when that is necessary,*

- *conservation of the support of functional surfaces (plans and cylinders in particular).*

V. Limitations and councils of use

Plate manages only specific constraints. In this strict context, the risks of failure are weak. There is incompatibility of constraints only if 2 contradictory constraints are given to the same parameter (U, v) (calling it will have to avoid imposing two specific constraints on the same order for two parameters (U, v) too much "close").
Apart from this case, the algorithm will give a solution if the condition of unicity is checked.
The condition of unicity (called "condition U" above) will have few chances to be violated apart from the cases described in chapter A below.

The difficulties appear when one wants to impose linear constraints (i.e in any point of one or more curves 2D of parametric space) and that one wishes to carry out a polynomial approximation of the solution given by the algorithm. Of course that corresponds in the majority of the cases of use and in particular of the applications referred to above.

A. "Plausible" case of nonrespect of the condition U

When number N of constraints is lower or equal to m (m+1)/2, the condition U is not checked. Indeed, the research of the polynomial not no one which invalidates the condition U consists in finding a solution nonnull of a homogeneous linear system having more unknown factors than of equations.

One is interested in the more important case (and more "typical" of the applications of Plate) where N is definitely higher than m (m+1)/2.
In this case, if the constraints were distributed by chance, the probability of nonrespect of U would be null and the probability of being close to such a case, very weak.
In practice the condition will be probably violated only when the constraints by construction are located on arcs of right-hand side or the conical ones of the parametric field.

For example, if the constraints are only C0 and located on a triangle of the parametric field, then the condition U will be checked only for m<4.
Indeed, there is a polynomial not no one of degree 3 (produced implicit equations of the right-hand sides supports of with dimensions of the triangle) which is cancelled on the edge of the triangle, and invalid the condition U for any higher value of m or equalizes to 4.
If one moreover wishes to impose C1 constraints on each edge, then there is a polynomial of degree 6 (the square of the precedent) which invalidates the condition U for m>6.

More generally, if the constraints come from the sampling of edges located on p segments of right-hand sides and Q arcs the conical ones, if one calls:

- *Odi the maximum order of constraints on the ième segment of right-hand side, (0 for a C0 constraint, 1 for a C0 constraint and C1, etc…)*
- *Oci the maximum order of constraints on the ième arc of conical,*

Then, there is a polynomial not no one of degree D, given by:

$$d = \sum_{i=1}^{p} (Od_i + 1) + 2\sum_{i=1}^{q} (Oc_i + 1)$$

obtained by product of the implicit equations corresponding to the supports raised with the power "ordre+1" which invalidates the condition U as soon as m >d.

This condition was applied in the use of Plate for H-PRISM (in this case there are only sampled segments of right-hand side) in order to limit the value of m supérieurement (independently of the terminal practices 6 or 7 due to the problems of numerical conditioning).

B. Forced linear

In order to impose a linear constraint, the user of the algorithm must sample the curve 2D and force the corresponding specific constraints.

He was shown (cf AD INTERIM Rozhenko and V.A. Vassilenko, Variationnal Approach in Splines Abstract: Achievements and Open Problems, Novosobirsk Computing Center off Siberian Branch off Russian Academy off Sciences, 1995) that, realizing certain assumptions of regularity and compatibility of the imposed linear constraints, the solution of the problem for sampled specific constraints converges (for a certain topology.) towards the solution of the problem with linear constraint continues when the step of sampling tends towards 0.

It would be theoretically possible to establish one raising of the error making it possible to know a step of sufficient sampling a priori to ensure that the error compared to the theoretical solution (i.e minimum under constraint continues) is lower than the desired value. Still it for that would be necessary to sample the linear constraint as a preliminary to deduce some from them raising… This method would lead to a too fine sampling (pessimist).

One advises a "pragmatic" approach very.
One samples with N points, where N is a in experiments regulated value into hard (N of about 100 for the whole of all the sampled curves for example). One measures then by sampling between the imposed points (1, 2 or 3 points per interval.) the order of magnitude of the made maximum error.
If that is not appropriate one starts again after having multiplied N by A, (>1 has) and one buckles on this treatment until satisfaction of the precision. This enrichment can be total or local. A local enrichment would consist is to be enriched only sampling by the curves on which the precision was not reached or to enrich even more locally (on portions of curves) according to the measured error.

If contour 2D is quite regular (i.e without corner and slightly curved), a sampling with constant curvilinear step in 2D is advised. On the other hand if contour contains corners, it is more effective to refine the step close to the corners (cf the report/ratio of training course PFE ENSAM Lille 96) for example by a distribution in "cosine" along each curve.
Such a distribution of the sampled points (as a cosine) should be used with a number of points per curve proportional to the square root length of the curves in order to be intervals comparable length on both sides of each corner.

Finally it is probably more effective to concentrate sampling in the zones of strong curve of the curve 2D. Since, often, the parameter of a curve "is concentrated" in the zones with strong curve, a sampling with constant parametric step is not inevitably worse than a sampling with constant curvilinear step.

C. Problems of incompatibility enters the constraints

As soon as linear constraints are imposed, several cases of incompatibility of constraints can occur. The algorithm will be able to give bad results because the data are close to a case of incompatibility.

The Plate users will have to keep in mind the important following rule:

With the order m, Plate can theoretically "absorb" only discontinuities of a nature equal to or higher than Gm-3 between two adjacent edges.

Thus, if one uses Plate with order 4, if one samples in a sufficiently fine way around a corner which has a discontinuity G2 (i.e discontinuity of curve or incompatibility of the type "twist vector"), the result of Plate can be unforeseeable and unacceptable.

The reason is that a specific discontinuity of order m2 (or inferior) creates in theory an infinite energy EM which can in practice have a total and unforeseeable effect on the result.

The first two problems below are the cause of the majority of the failures of the algorithm of correction of the hulls.

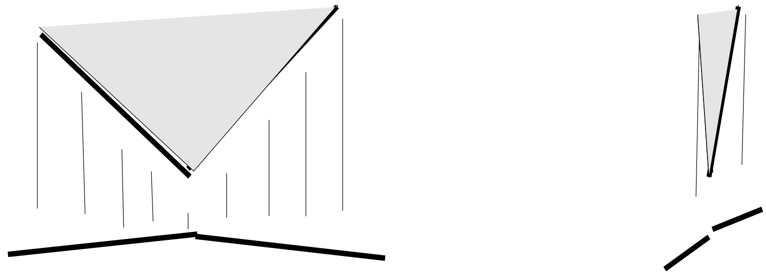
1. Superimposed curves 2D.

This case corresponds to the case of incompatibility between incompatible specific constraints in the same parameter.
A curve 2D is folded up on itself or several curves 2D corresponding to linear constraints are superimposed or intersected by imposing different constraints on the same parameters (U, v).

2. Curves 2D tangents and forced not "tangents"

If two successive curves 2D in contour are tangent, their compositions with the function solution (which is at least C1 and often more.) are necessarily tangent, in the direction where their derivative owe beings proportional in the same report/ratio as those of the curves 2D tangents. There is thus incompatibility of the linear constraints if it is not the case. This incompatibility will result in a "distortion" of the all the more large solution as the step of sampling will be finer.

If, finally, the two successive curves are "quasi-tangents" and that the specified values are not it, the solution will present a strong "slope" as shown in the figure following.



3. Incompatibility of corner of the type "twist vector"

It is known that G1 compatible constraints in tangency applied in a corner can induce incompatibilities on one of the components of the curve of forced surface (the "twist vector").

G2 compatible constraints in curve can induce incompatibilities on characteristics of a higher nature of forced surface. These incompatibilities make impossible a filling by a regular polynomial surface since such a surface is continuous with all the orders (G¥).
These incompatibilities can be formulated in terms of constraints C1 or C2: to satisfy certain C1 constraints in a corner, the function solution cannot satisfy the symmetry of derived the second cross ones (equality from Schwartz) and cannot thus be 2 times derivable at the corner.
The difficulty which arises here is that one can have (by choosing a rather weak order m) a function solution which satisfies these "incompatible" constraints and which is thus not 2 times derivable into 1 point (in the case of incompatibility between G1 constraints) but one will not be able, downstream from the algorithm, to correctly approach this function by a polynomial.

From a practical point of view, the experiment shows that the polynomial approximation is often acceptable, insofar as approximate surface "violates" the constraints very locally, in the vicinity of the incompatible corner, and gives a visually satisfactory result then (including after control of the lines of light).

Perhaps is it conceivable to carry out an approximation by a rational function whose denominator S "cancels with the corners or, after a singular change of variable to the corners, by a polynomial function. For L" urgent, that remains to be shown. Perhaps work of Olivier Gibaru (thesis CIFRE Scientific department in progress) will bring elements to us on the question.

VI. Plate extensions future

Plate is only one example simple of energy method. We are convinced that an investment in these methods would make it possible to confirm and to develop our technological advance in the field of the style and complex surfaces.

A. Automatic determination more "fine" of an initial surface

Currently, the "filling with N with dimensions" coded in STRIM uses an average plan like surfaces initial. It seems that "quadric less" squares calculated starting from the plan of maximum flow (in the case of a closed contour) constituerait an initial surface which would often give a better result and would decrease the cases of failures.
This "extension" should be taken into account in an operator CAS.CADE of higher level which would include Plate.

B. Plate Extensions based on the same numerical method

1. Specific "Loadings" and in pressure, "dynamic" mode

This extension is inexpensive. It would make it possible to propose interactive actions of deformation of surface spectacular.

2. smoothing

The method suggested for the interpolations can spread to operate smoothings. Thus, in addition to the "exact" constraints one could add constraints "to be approached" with a weight allowing "to balance" (this weight could be automatically given (repeatedly) according to an error of maximum smoothing).

3. forced inequality

One could manage constraints of inequality (by equations of the Kuhn-Tucker type). This extension asks for a phase of research (especially of prototyping and test, because the convergence of the method of incremental loading with management of the active constraints remains to be proven).

One can imagine like application which one asks to create a “casing” obeying constraints on the edge and in front of “including” a zone of obstruction given. The condition of noncollision with the zone of obstruction would result in inequalities, which would be treated repeatedly.

4. A polynomial approximation adapted to Plate

The tool for polynomial approximation one has is not adapted to the surfaces generated by Plate. The same engine which would make it possible to create polynomial surfaces splines by minimization under constraints (cf method of Ritz below) would allow an effective polynomial approximation because able to differently balance the error of approximation on the edges (nonisoparametric!) and on the “medium” of surface.

In addition, it could interesting, like be announced higher, to manage a change of variable making it possible to generate singular surfaces when one knows a priori the points which one wishes singular. The automatic determination of this change of variable can be rather difficult to carry out when the parametric field is complex (returning corners for example).

C. Plate Extensions based on a method of Ritz (finite elements)

1. Linear criteria

If the same criteria are preserved, a method of Ritz would make it possible to directly obtain the result approached in polynomial form and would be conditioned better, which would make it possible to reach in all the cases the desired precision, which is not the case of Plate today. This development is inexpensive and not very risky.

2. Nonlinear criteria

The installation of an engine for the polynomial approximation or the application of the method of Ritz applied to the linear criteria should make it possible to test an algorithm of the Newton-Raphson type for the minimization of geometrical criteria (i.e invariants by reparametrisation) and thus non-linear. That could also constitute an algorithmic base for the design by objective applied to the style (cf European project FIORES) like for the variational design applied to complex surfaces. A “variational surface” would be a surface defined by a criterion and constraints, able to react in a foreseeable way (by preserving esthetic criteria in particular.) with a modification of the geometrical entities by report/ratio to which the constraints are defined.

This field is in our promising but rather risky opinion, in the direction where it asks a significant part of research and prototyping before giving place to an industrial application.

[1]That supposes that one has a topology enriched by information by continuity G1 and G2 on the edges and the tops. In addition the correction must, as far as possible, preserve the supports of simple surfaces (plane cylinders, spheres.).

[2]*With a multiplicative coefficient close depend on the thickness of the plate and the Young modulus of material constituting it.*

[3]*The concept of "regular" parameter setting is intuitive but corresponds to the fact that the metric one (i.e the first fundamental form) varies slightly, which supposes that surface too is not far from a developable surface.*

[4]*That does not mean that this local effect is a consequence of the algebraic character of these methods.*

[5]*Xm DM (L2 (R2) is in general noted), and can be provided with a "natural" structure of space of Hilbert (see for example Atteia 92, pages 172 and 173 for more details).*

[6]*In any rigor, this derivative is defined only if the quadratic form is continuous, which is the case here. cf appendix 1.*

[7]*cf appendix 1.*

[8]*That appears for example in Atteia92, page 174.*

[9]*It will be admitted that the second part of proposal 4.3, pages 173 and 174 of Atteia92, spreads if μ is a distribution with compact support which is prolonged in a continuous form linear on Xm (what is the case of the second member of ED bus ik+jk£m-2).*
The condition:

$$\forall p \in P_{m-1},\ \mu(p) = 0$$

proposal 4.3 of Atteia92 is carried out by the first form of the balance of generalized moments EMG1. cf appendix 1 for more details.

[10]*That can be shown for example from [Lelong-Ferand 65, differential Géométrie, page 121 "contact of order R", at Masson], where, in a more elementary way, by building quadric osculatory*